Electric potential

Potential energy and path integrals; conservative force

Work done by the gravitational field of a point mass

If we have an object of mass m moving in the gravitational field of a point mass M, we can calculate the work done by the gravitational field as mass m moves from point A to point B.

$$W_g = \int_A^B \mathbf{F}\cdot\mathbf{ds} = \int_{r_A}^{r_B} \frac{GMm}{r^2}\,dr = -\frac{GMm}{r_B} - \frac{-GMm}{r_A}$$

We see that this integral depends only on the initial and final position of the object, and in particular it does not depend on the path. It follows that over any closed path, the work done by gravity will be zero, and therefore we can define a unique function that we call the potential energy. The change in gravitational potential energy when an object moves from a point A to a point B is *defined* as the negative of the work done by gravity.

$$\Delta U = -W_g = -\int \mathbf{F}\cdot\mathbf{ds}$$

If we think about out two objects as a system, then to move the one object, we need to have some outside force do work, exactly equal in magnitude to the work done by gravity, to make this happen. The change in energy of our system is then equal to the work done by the outside force.

Only changes in potential energy are meaningful, so we are free to define the zero of potential energy at any point. Frequently, we will choose to set U=0 when the masses are infinitely far apart, in which case:

$$U(r) = \frac{-GMm}{r}$$

Charge moving in a uniform electric field

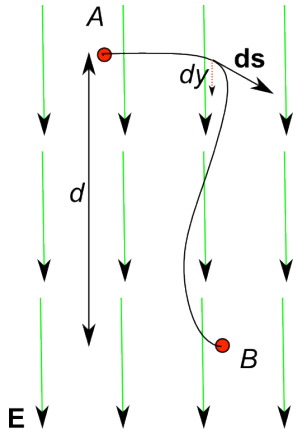
If we have an object of charge q in a uniform electric field and we move it from point A to point B, then the work done by the electric field will be given by:

$$W_e = \int_A^B q\mathbf{E}\cdot\mathbf{ds}$$

which, for a uniform field, becomes:

$$W_e = \int_A^B qE\,dy = qEd$$

where dy is the component of **ds** along the direction of the electric field and d is the component of the total displacement of the charge in the direction of the electric field.



Once again, the work does not depend on the path, so we define the electric potential energy:

$$\Delta U = U_B - U_A = -W_e = -\int_A^B q\mathbf{E}\cdot\mathbf{ds}$$

For a uniform electric field:

$$\Delta U = -qEd$$

Note that d can be positive or negative, and q can be positive or negative. If we move a positive charge along the electric field, the potential energy decreases, whereas if we move it opposite the field, the potential energy increases. The exact opposite holds for a negative charge.

Definition of potential

When we calculate the change in potential energy when we move a charge, we notice that it is proportional to the amount of charge moved, so that if we moved an object with twice the charge, the change in potential energy would be twice as great. It makes sense, therefore, to define a change in "potential energy per unit charge" between two points, which is the change in potential energy the results from moving a charge from one point to the other divided by the amount of charge.

$$\Delta V = V_B - V_A = \frac{\Delta U}{q} = -\int_A^B \mathbf{E}\cdot\mathbf{ds}$$

Because the zero of potential energy is arbitrary, we are free to set the point of zero potential wherever we would like.

The units of electric potential is joules per coulomb, and 1 V (volt) = 1 J/C

With this definition, at points of higher potential, positive charges will have *more* potential energy and negative charges will have *less* potential energy.

The relationship between electric potential and electric forces is similar to the relation between elevation and gravity. If we move an object to higher elevation, its potential energy will increase in proportion to its mass. If we move a positive charge to a position of higher potential, its potential energy will increase in proportion to its charge.

Calculating electric potential

Uniform electric field

The potential difference between two points in a uniform electric field is:

$$\Delta V = -Ed$$

where d is the component of the displacement $\mathbf{\Delta r}$ from A to B ($\mathbf{\Delta r}=\mathbf{r}_B-\mathbf{r}_A$) in the direction of $\mathbf{E}$. Sometimes the sign of the potential difference can become confusing. We always have to remember that positive charges have greater potential energy at points where the potential is greater, so that potential always increases as we move *upstream* in an electric field.

Potential near a point charge

We can calculate the potential difference between two points around a point charge:

$$\Delta V = V_B - V_A = -\int_A^B \mathbf{E}\cdot\mathbf{ds} = -\int_{r_A}^{r_B} \frac{q}{4\pi\varepsilon_o r^2}\,dr$$

$$\Delta V = \frac{q}{4\pi\varepsilon_o r_B} - \frac{q}{4\pi\varepsilon_o r_A}$$

If we take the potential to be zero infinitely far from the point charge, then we can write the potential around a point charge as:

$$V_B = -\int_\infty^B \mathbf{E}\cdot\mathbf{ds} = \frac{q}{4\pi\varepsilon_o r_B}$$

Superposition and potential near several point charges

If we have a collection of point charges, because of superposition the total potential is equal to the sum of the potentials from each point charge.

We can use this to calculate the potential energy of a distribution of charges. Let say we have 3 charges q_A, q_B, and q_C at positions $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$. We will calculate the potential energy by calculating the work required to bring in each from infinitely far away.

1. Bring in q_A. ΔU=0

2. Bring in q_B. $\Delta U = q_B V_{AB}$, where V_{AB} is the potential at $\mathbf{r}_B$ due to q_A.
3. Bring in q_C. $\Delta U = q_C V_{AC} + q_C V_{BC}$

Therefore:

$$U = q_B V_{AB} + q_c V_{BC} + q_c V_{AC} = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_A q_B}{r_{AB}} + \frac{q_B q_C}{r_{BC}} + \frac{q_A q_C}{r_{AC}}\right)$$

Electric potential due to a continuous charge distribution

If the charge distribution is continuous rather than discrete, then we need to add up all the contributions to the potential from each infinitesimal element of charge dq.

$$dV = \frac{dq}{4\pi\varepsilon_o r}$$

$$V = \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{r}$$

We will find that one of the advantages of using potential is that it is a scalar!

Examples/Problems

Four charges on a square: Consider four point charges of charge $+q$ that are arranged on the corners of a square with side a. Determine the electric potential at the center of the square. What is the electric field at the center of the square? Sketch the potential versus distance as you move away from the center of the square in a direction i) along a diagonal; ii) perpendicular to one side. Is the center of the square an equilibrium point?

Electric dipole

Consider an electric dipole consisting of charges $+q$ and $-q$ separated by a distance $2a$ on the x-axis. Calculate the electric potential as a function of x on the x-axis. What is the potential on the y-axis? See http://web.mit.edu/8.02t/www/802TEAL3D/visualizations/coursenotes/modules/guide03.pdf section 3.8.1.

Parallel-plate capacitor.

Two parallel plates of area A carrying charge $+Q$ and $-Q$ and area are separated by a distance d. Assume that the left plate is positively charged, as shown in the figure. Determine the electric field between the two plates and the potential difference between the plates. Assume that the separation of the plates is much less than the dimensions of the plates

Solution Let's start by assuming the plates are insulators. They carry a uniform charge per unit area $\sigma = Q/A$. We can calculate the electric field between the plates using Gauss's Law and superposition. The field due to each plate is:

$$\mathbf{E}_{left} = \mathbf{E}_{right} = \frac{\sigma}{2\varepsilon_0} \text{ to the right}$$

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0} \text{ to the right}$$

Note that our answer would be the same if the plates were conductors. In this case, each surface of the conductor would have a surface charge density of $\sigma/2$.

It is also interesting that we used Gauss's law to calculate the contribution to the field from each plate. However, Gauss's law holds generally, so that if I draw a pill-box through the surface one of the conductors (the left one, to be specific), the total flux through the enclosed surface must equal the enclosed charge divided by ε_0. This works since bringing the right hand plate in doubles the field to

the right of the left hand plate, but cancels the field to the left of the left-hand plate. Isn't physics amazing!

Since the field between the plates is uniform, the potential difference between them is just:

$$\Delta V = -\int \mathbf{E}\cdot\mathbf{dl} = Ed = \frac{\sigma d}{\varepsilon_0} = \frac{d}{\varepsilon_0 A}Q$$

Since the left hand plate is positively charged, it will be at the higher potential.

Parallel plate capacity with conducting slab inserted.

Consider the case of a conducting slab of thickness d_m<d placed midway between the capacitor plates. Because of the electric field of the outside plates, electrons on the conducting slab will migrate in such a way that the total electric field inside the conductor is zero (see figure), and a charge of $+Q'$ and $-Q'$ will reside on the surfaces of the conductor. In order for the field inside the conductor to be zero, however, the amount of charge Q' on the surface of the conductor must be equal to the charge on each capacitor plate. A simple argument based on superposition will convince of this. We are left, therefore with the space between our capacitor plates divided into three regions. In the two regions where the conductor is the electric field is zero, and in the regions on either side of the conductor, the field is exactly what it was before introducing the conducting slab. The potential difference between the two plates is therefore:

$$\Delta V = \frac{\sigma}{\varepsilon_0}(d - d_m) = \frac{(d - d_m)}{A\varepsilon_0}Q$$

With the conducting slab between the plates, then, for the same potential difference, more charge will reside on the plates than with no conducting slab.

Charged rod

A charged rod with charge Q and length L lies along the x-axis, and extends from x=-l/2 to x=+l/2. Calculate the electric potential along as a function of distance from the rod along the y-axis. See http://web.mit.edu/8.02t/www/802TEAL3D/visualizations/coursenotes/modules/guide03.pdf example 3.1.

Electric potential around a charged sphere.

Using Gauss's law, we previously determined that the electric field outside a uniformly charged sphere of radius R with charge Q is equal to :

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

Taking V=0 infinitely far from the sphere, we can calculate the potential some distance r>R.

$$V(r) = -\int_{-\infty}^{r} \mathbf{E}\cdot\mathbf{dl} = -\int_{-\infty}^{r} \frac{Q}{4\pi\varepsilon_0}\frac{dr'}{r'^2} = \frac{Q}{4\pi\varepsilon_0 r} \quad r > R$$

If the sphere is conducting, then inside the sphere the electric field is zero and the potential will be constant and equal to its value at the surface:

$$V(r) = \frac{Q}{4\pi\varepsilon_0 R} \quad r \le R$$

In the case of an insolating sphere, however, we showed earlier that the electric field inside the sphere is:

$$E = \frac{Qr}{4\pi\varepsilon_o R^3} \text{ for } r < R$$

The potential inside the sphere is:

$$V(r) = V(R) - \int_R^r \mathbf{E}\cdot\mathbf{dl} = \frac{Q}{4\pi\varepsilon_0 R} - \int_R^r \frac{Q}{4\pi\varepsilon_0 R^3} r\,dr = \frac{Q}{4\pi\varepsilon_0 R} - \frac{Q}{4\pi\varepsilon_0 R^3}\left[\frac{r^2}{2} - \frac{R^2}{2}\right]$$

$$V(r) = \frac{3Q}{8\pi\varepsilon_0 R} - \frac{Qr^2}{8\pi\varepsilon_0 R^3} = \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) \quad r \le R$$

Electric potential difference between two charged spheres

Calculate the potential difference between the surfaces of two identical spheres of radius R with a center to center spacing d and charge $+Q$ on one and $-Q$ on the other. We can calculate the potential difference by taking the integral of **E·dl** along the line connecting the centers of this sphere.

$$\Delta V = -\int_R^{d-R} E\,dr = -\int_R^{d-R}\left[\frac{Q}{4\pi\varepsilon_0 r^2} + \frac{Q}{4\pi\varepsilon_0 (d-r)^2}\right]dr = \frac{-Q}{4\pi\varepsilon_0}\left[\int_R^{d-R}\frac{dr}{r^2} + \int_R^{d-R}\frac{dr}{(d-r)^2}\right]$$

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left[\left(\frac{1}{d-R} - \frac{1}{R}\right) - \left(\frac{1}{d-(d-R)} - \frac{1}{d-R}\right)\right] = \frac{2Q}{4\pi\varepsilon_0}\left(\frac{1}{d-R} - \frac{1}{R}\right)$$

Electric potential difference between a sphere of charge $+Q$ and radius a and a concentric spherical surface of charge $-Q$ and radius $b>a$. The electric field between the surfaces is:

$$\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \quad a < r < b$$

and the potential difference is given by:

$$\Delta V = -\int_a^b E\,dr = -\frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$$

Note that if b is infinitely large and we take V=0 on the shell, then the potential on the surface of the sphere is:

$$V = \frac{Q}{4\pi\varepsilon_0 a}$$

which is the same result we got earlier. Isn't physics beautiful?!

Implications of a conservative force on the mathematical nature of the electric field

We have shown that for the field created by a single point charge, the change in potential between two points, which we define as

$$\Delta V = \int_A^B \mathbf{E}\cdot\mathbf{dl}$$

is independent of path and therefore zero on any closed path. Furthermore, from the principle of superposition we conclude that the above integral is path independent for a field that results from any combination of stationary point charges.

However, imagine the case of an electric field described by:

$$\mathbf{E} = \frac{y}{x^2+y^2}\hat{\mathbf{i}} - \frac{x}{x^2+y^2}\hat{\mathbf{j}}$$

This is the field of circulation pattern. If you start at the point (1,0) and integrate **E·dl** around the unit circle you will <u>not</u> get zero. Therefore, we conclude that it is impossible to get this electric field from any arrangement of stationary charges. However, we will see that a *non-conservative* electric field of this mathematical form can arise as a result of other circumstances.

Electric field and potential gradient

Equipotential surfaces

For any charge distribution, we can identify equipotential surfaces on which the electric potential is constant. See http://hyperphysics.phy-astr.gsu.edu/hbase/electric/equipot.html#c2 for three simple examples:

- In the case of parallel plates with charge $+Q$ and $-Q$, the potential is a function only of the distance from one of the plates, so that any plane parallel to the two plates would be an equipotential surface.
- In the case of a spherical charge distribution, any spherical surface concentric with the spherical charge distribution would be an equipotential surface.
- For a more complicated arrangement, such as a dipole, the equipotential surfaces are more difficult to describe simply. One equipotential surface around a dipole would be the plane that is everywhere equidistance from the two charges.

An equipotential surface is to electric fields as a topographic map is to a gravitational field.

The electric field at every point is perpendicular to the equipotential surface passing through that point. If a charged particle has a displacement **ds** along an equipotential surface, its potential energy doesn't change. Therefore, the electric field does no work on the particle, and the only way this can be true is if $\mathbf{E}\cdot\mathbf{ds}=0$, which implies that **E** is perpendicular to the surface.

The strength of the electric field is determined by the rate with distance that the potential changes. In the simple case of a uniform field:

$$\Delta V = -Ed$$

$$E = -\frac{\Delta V}{d}$$

where d is the displacement along the direction of the field. We can generalize this:

$$dV = -\mathbf{E}\cdot\mathbf{ds} = -E ds_{\perp}$$

$$E = -\frac{dV}{ds_{\perp}}$$

where $ds_{\perp}$ is a displacement perpendicular to the equipotential surface (and parallel to **E**).

In summary: **E** at any point is perpendicular to the equipotential surface passing through that point, it points in the direction that V is decreasing, and its magnitude is the slope of the potential vs distance as we move perpendicular to the surface. Note also that the direction of **E** is also the direction the V decreases most steeply.

Defining the gradient of the potential; partial derivatives

In more advanced courses, you will write this analytically as:

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right) = -\nabla V$$

where ∇ (pronounced "del") is called the gradient operator. The partial derivatives, for example $\frac{\partial V}{\partial x}$ means we take the derivative of V with respect to x, while keeping y and z constant (move in a path in the $+x$ direction).

Examples/problems

Calculate the field and potential, and show that $\mathbf{E} = -\nabla V$ for the following cases.

On the axis of a dipole for |x|>>a

Consider an electric dipole with a charge $-q$ at $x=-a$ and charge $+q$ at $x=+a$. Calculate the potential and electric field at a point $x>+a$, and show that

$$E_x = -\frac{dV}{dx}$$

Show also that for $x>>a$ the field is proportional to $1/x^3$ and the potential is proportional to $1/x^2$.

The electric potential for $x>a$ is given by:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{x-a}+\frac{-q}{x+a}\right) = \frac{1}{4\pi\varepsilon_0}\frac{2aq}{\left(x^2-a^2\right)}$$

The electric field for $x>a$ is in the $+x$ direction and we calculate:

$$E_x = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}+\frac{q_2}{r_2^2}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\left(x-a\right)^2}+\frac{-q}{\left(x+a\right)^2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{4aqx}{\left(x^2-a^2\right)^2}$$

To show that the field is the negative of the gradient of the field, we take the derivative of the potential with respect to x (since we can argue from the geometry that the field must be in the x direction.

$$E_x = -\frac{dV}{dx} = -\frac{2aq}{4\pi\varepsilon_0}\frac{d}{dx}\left(x^2-a^2\right)^{-1} = -\frac{2aq}{4\pi\varepsilon_0}\left(\frac{-2x}{\left(x^2-a^2\right)^2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{4aqx}{\left(x^2-a^2\right)^2}$$

And this is the same result that we got before. In applications of field theory, it is often easiest to perform calculations using the potential, and then, if needed, the vector field can be obtained from the potential.

In the case of dipoles it is also important to see how that field and potential behave for $x>>a$.

$$V = \frac{2aq}{4\pi\varepsilon_0}\frac{1}{x^2-a^2} \approx \frac{2aq}{4\pi\varepsilon_0 x^2} \quad \text{for } x >> a$$

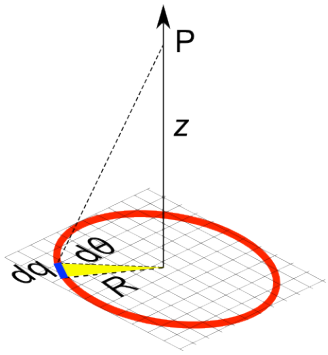
$$E_x = \frac{4aq}{4\pi\varepsilon_0}\frac{x}{\left(x^2-a^2\right)^2} = \frac{4aq}{4\pi\varepsilon_0}\frac{x}{x^4\left(1-\frac{a^2}{x^2}\right)^2} \approx \frac{4aq}{4\pi\varepsilon_0 x^3} \quad \text{for } x >> a$$

And it is also easy to see from these results that

$$E_x = -\frac{dV}{dx}$$

<u>A ring of charge</u>

Consider the case of a ring of charge, shown in the figure, lying in the xy (horizontal) plane, and we want to find the electric field and potential at any point on the axis of the ring (z axis) a distance z from the plane of the ring. The ring has a radius R, a total charge Q, and we assume that the ring is thin and can be treated as a line of charge.



To calculate the electric potential at point P, we need to break the ring into infinitesimal segments of charge dq. In the drawing dq is shown by the small blue portion of the ring. The total potential at P is then given by:

$$V = \frac{1}{4\pi\varepsilon_0}\int_{ring}\frac{dq}{r}$$

where dq is the charge of the infinitesimal segment, colored in blue in the figure, and r is its distance from point P.

At this point, we really haven't done anything except restate the principle of superposition: that the potential at any point is equal to the sum of the potentials due to each point charge near that point. The integral is just a notation we use for a summation when we have a continuous distribution of charges and we split it up into infinitesimal pieces. Writing the above equation, therefore, is the easy part. The challenge always comes in figuring out how to turn the integral into something that can be calculated.

The first step in this process is usually to decide on the variable of integration to use, and the geometry of the situation usually provide a hint. Since we are integrating around a circle, it is natural to define a variable θ, and we can integrate around the entire ring by integrating from θ=0 to θ=2π. We can now write dq in terms of our variable of integration. Since dq is equal to the linear charge density, λ, times the length of the infinitesimal segment:

$$dq = \lambda R d\theta$$

where dθ is the infinitesimal angle colored in yellow in the figure. The distance, r, between the element of charge and point P is just

$$r = \sqrt{z^2 + R^2}$$

The integral above becomes:

$$V = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{\lambda R d\theta}{\sqrt{z^2+R^2}}$$

Everything in the integrand, however, is constant so this becomes simply

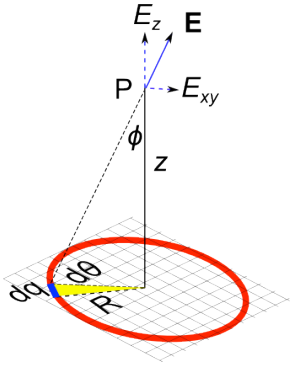
$$V = \frac{\lambda R}{4\pi\varepsilon_0\sqrt{z^2+R^2}}\int_0^{2\pi} d\theta = \frac{2\pi R\lambda}{4\pi\varepsilon_0\sqrt{z^2+R^2}} = \frac{Q}{4\pi\varepsilon_0\sqrt{z^2+R^2}}$$

This is not a surprising result since very point on the ring is equidistant from point P.

We have two methods now that we can use to calculate the electric field. First, however, it is useful to note that the total electric field at P will be in the z direction, because the component of the electric electric field from our charge element dq that perpendicular to the z direction (i.e., in the xy plaen; denoted E_{xy} in the figure) will cancel exactly with that component of the field from the element on the opposite side of the ring, and this cancellation will occur for every point on the ring.

Since the electric field is in the y direction, it will be equal to:

$$\mathbf{E} = -\frac{dV}{dz} = -\frac{d}{dz}\left[\frac{Q}{4\pi\varepsilon_0}\left(z^2+R^2\right)^{-1/2}\right]\hat{\mathbf{k}} = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{-1}{2}\right)2z\left(z^2+R^2\right)^{-3/2}\hat{\mathbf{k}} = \frac{Qz}{4\pi\varepsilon_0}\left(z^2+R^2\right)^{-3/2}\hat{\mathbf{k}}$$



Now, it would be nice to show that we get the same result when we calculate the electric field using Coulomb's law. In this case:

$$E_z = \frac{1}{4\pi\varepsilon_0}\int_{ring}\frac{dq}{r^2}\cos\phi$$

which we can evaluate by taking the integral of θ around the ring.

$$E_z = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{\lambda R\cos\phi d\theta}{\left(z^2+R^2\right)} = \frac{\lambda R}{4\pi\varepsilon_0}\frac{1}{\left(z^2+R^2\right)}\frac{z}{\sqrt{z^2+R^2}}\int_0^{2\pi}d\theta = \frac{Q}{4\pi\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{3/2}}$$

This is exactly the same result as we got by calculating the field from the potential!

A disk of charge

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by Stephen Palfrey 2011-02-05